1 Ordinary Generating Functions

This section is based originally on Chapter 6: Generating Functions of *Inquiry-Based Enumerative Combinatorics* by T. Kyle Petersen. Warmups and problems have been added or atomized to build understanding more intuitively, and added discussion aims to make connections to other mathematical disciplines you might have encountered already. At any level of familiarity with power series or indeed generating functions, it is quite beneficial to work through each warmup and problem in detail: it is the particulars of these objects that make them so unreasonably effective. If a feature or application of ordinary generating functions feels clunky, you might ask yourself ...

"What would a generating function need to look like for this to work smoothly?"

Chances are, there is an extraordinary type of generating function which behaves nicely in that sense. This should raise additional questions about *why* generating functions with this feature would be of any use. The answers to those questions are a lot of fun to stumble across yourself, so let's start walking and see what we find.

1.1 Properties of generating functions

Definition 1.1 (Ordinary generating function)**.** Given a sequence of numbers $(a_k)_{k\geq 0} = (a_0, a_1, a_2, \ldots, a_k, \ldots)$, we define its ***formal power series*** by:

$$F(z) = \sum_{k\geq 0} a_k z^k = a_0 + a_1 z + a_2 z^2 + \cdots + a_k z^k + \cdots \ .$$

We also refer to F as the ***(ordinary) generating function*** for the sequence.

Notation. By convention,

- capital letters F, G, A, etc. are used to denote generating functions
- late-alphabet lowercase letters q, t, x, y, z, etc. are used for arguments/formal variables (but q is extra special)
- mid-alphabet lowercase letters i, j, k, ℓ, m, n are used for indices

Proposition 1.2 (Properties of formal power series)**.**

(i) $c \cdot \sum_{k\geq 0} a_k z^k = \sum_{k\geq 0} c a_k z^k$ *for any constant* c

(ii) $\left(\sum_{k\geq 0} a_k z^k\right) + \left(\sum_{\ell\geq 0} b_\ell z^\ell\right) = \sum_{m\geq 0} (a_m + b_m)\, z^m$

(iii) $\left(\sum_{k\geq 0} a_k z^k\right)\left(\sum_{\ell\geq 0} b_\ell z^\ell\right) = \sum_{j\geq 0}\left(\sum_{k+\ell=j} a_k b_\ell\right) z^j = \sum_{j\geq 0}\left(\sum_{k=0}^{j} a_k b_{j-k}\right) z^j$

Warmup 1. Convince yourself of part (iii) of Proposition 1.2 the long way. As a starting point, we can rewrite $\left(\sum_{k\geq 0} a_k z^k\right)\left(\sum_{\ell\geq 0} b_\ell z^\ell\right)$ as

$$\left(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots\right)\left(b_0 + b_1 z + b_2 z^2 + b_3 z^3 + b_4 z^4 + \cdots\right) .$$

Carefully write out the multiplication of the series to include all terms with powers of z up to 4, then group terms according to the power of z.

Note. We can think of finite sequences as infinite sequences that just have an infinite "tail" of zeros at the end. As a result, the equation in the Binomial Theorem can also be viewed as a formal power series

$$F(x) = \sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n$$

Problem 1. Using the power series perspective, as above, for $(1+x)^m$ and $(1+x)^n$, find two forms of the coefficient of x^ℓ in the polynomial $(1+x)^{m+n}$.

Remark. Computing derivatives works too!

$$\begin{aligned} F'(z) = \frac{\mathrm{d}}{\mathrm{d}z}\left[\sum_{k\geq 0} a_k z^k\right] &= \sum_{k\geq 0} \frac{\mathrm{d}}{\mathrm{d}z}\left[a_k z^k\right] && \text{by the linearity of the derivatives} \\ &= \frac{\mathrm{d}}{\mathrm{d}z}\left[a_0\right] + \sum_{k\geq 1} \frac{\mathrm{d}}{\mathrm{d}z}\left[a_k z^k\right] && \text{isolating the first term} \\ &= 0 + \sum_{k\geq 1} a_k k z^{k-1} && \text{classic power rule} \\ &= \sum_{\ell\geq 0} a_{\ell+1}(\ell+1) z^\ell && \text{re-indexing: } \ell + 1 \doteq k \end{aligned}$$

If you have taken some calculus, everything but the last step should ring some bells. If you haven't you can take the "classic power rule" step as a black box. The re-indexing step is the most likely to be unfamiliar, but is a commonly used tool in a combinatorialist's tool belt. Remember that in Definition 1.1, the sequence and power series begins with $k = 0$. We want $F'(z)$ to take this form as well, which means the lower bound of our sum must shift down from $k = 1$ to $\ell = 0$. However, if that is the only change we make, the two sums won't be equal. You should confirm for yourself that the shift within the summands of the expression above really do preserve equality.

1.2 Foundational generating functions and their sequences

The word ***formal*** in "formal power series" means that the variable in the power series is not intrinsically important to us as combinatorialists. Rather, the variable exists primarily to accompany the terms of the sequence $(a_0, a_1, a_2, \ldots, a_k, \ldots)$.

As you begin to understand ordinary generating functions here, and more extraordinary generating functions later, a good starting point is considering the generating function associated to the second-most boring sequence, $(1, 1, 1, \ldots)$. For ordinary generating functions, that is

$$F(z) = \sum_{k\geq 0} z^k = 1 + z + z^2 + z^3 + z^4 + \cdots$$

Warmup 2. Let $F(z) = \sum_{k\geq 0} z^k$ and $G(z) = 1 - z$. Annotate the work below.

$$
\begin{aligned}
&(1) \qquad F(z)G(z) = \left(\sum_{k\geq 0} z^k\right)(1-z) \\
&(2) \qquad = \left(1 + z + z^2 + z^3 + z^4 + \cdots\right)(1-z) \\
&(3) \qquad = \left(1 + z + z^2 + z^3 + z^4 + \cdots\right) - z\left(1 + z + z^2 + z^3 + z^4 + \cdots\right) \\
&(4) \qquad = 1 + z + z^2 + z^3 + z^4 + \cdots - z - z^2 - z^3 - z^4 - \cdots \\
&(5) \qquad = 1 + \cancel{z} + \cancel{z^2} + \cancel{z^3} + \cancel{z^4} + \cancel{\cdots} - \cancel{z} - \cancel{z^2} - \cancel{z^3} - \cancel{z^4} - \cancel{\cdots} \\
&(6) \qquad = 1
\end{aligned}
$$

The punchline of this work is that $F(z)$ can be defined nicely.

$$F(z) = \boxed{}$$

Problem 2. Taking the geometric series identity $\sum_{k\geq 0} z^k = \frac{1}{1-z}$ as a starting point, what are the **sequences** are defined by the generating functions below?

Heads up: Though a necessary first step is finding the series function below, that is not the end goal. You should finish the job by finding the **sequence** the series defines.

(a) $\frac{1}{1+z}$

(b) $\frac{1}{1-2z}$

(c) $\frac{1}{z-3}$

(d) $\frac{1}{1-z^2}$

(e) $\frac{1}{(1-z)^2}$

Problem 3. Multiplying a power series by $\frac{1}{1-z}$ has a nice effect on the power series.

If $F(z) = \sum_{j\geq 0} a_j z^j$, what sequence has generating function $\frac{F(z)}{1-z}$?

Hint: For this problem, you should use a go-to trick for generating functions. Make it clear that your given function really is a generating function with an equation, in this case $\frac{F(z)}{1-z} = \sum_{\ell\geq 0} b_k z^k$.

You're then looking for a way to express the numbers b_k in terms of the a_j.

Definition 1.3. The n^{th} triangular number, denoted T_n, is the sum of the first n natural numbers

$$1 + 2 + 3 + \cdots + n\ .$$

You might know by a piece of Gauss trivia, or by considering complete graphs or how two out of n friends could be allowed into a dance club by a bouncer, that $T_n = \frac{n(n+1)}{2}$. The sequence of triangular numbers is defined by the recurrence relation $T_n = T_{n-1} + n$ for $n \geq 1$, and $T_0 = 0$

Problem 4. Show that $\dfrac{1}{(1-z)^3} = \displaystyle\sum_{n\geq 0} T_{n+1} z^n$ using a familiar approach.

Hint: Part (e) of Problem 2.

Problem 5. This problem shows a new-to-you but classic approach to finding the same result as Problem 4. First, annotate the work below (work with either the expanded or the summation notation version):

$$
\begin{aligned}
&(1) \quad A(z) = T_1 + T_2 z + T_3 z^2 + T_4 z^3 + T_5 z^4 + \cdots \\
&(2) \quad\quad = (T_0 + 1) + (T_1 + 2)z + (T_2 + 3)z^2 + (T_3 + 4)z^3 + (T_4 + 5)z^4 + \cdots \\
&(3) \quad\quad = (T_0 + T_1 z + T_2 z^2 + T_3 z^3 + T_4 z^4 + \cdots) + (1 + 2z + 3z^2 + 4z^3 + 5z^4 + \cdots) \\
&(4) \quad\quad = (T_1 z + T_2 z^2 + T_3 z^3 + T_4 z^4 + \cdots) + (1 + 2z + 3z^2 + 4z^3 + 5z^4 + \cdots) \\
&(5) \quad\quad = z(T_1 + T_2 z + T_3 z^2 + T_4 z^3 + \cdots) + (1 + 2z + 3z^2 + 4z^3 + 5z^4 + \cdots) \\
&(6) \quad\quad = z(A(z)) + \frac{1}{(1-z)^2}
\end{aligned}
$$

$$
\begin{aligned}
&(1) \quad A(z) = \sum_{n\geq 0} T_{n+1} z^n \\
&(2) \quad\quad = \sum_{n\geq 0} (T_n + n + 1) z^n \\
&(3) \quad\quad = \sum_{n\geq 0} T_n z^n + \sum_{n\geq 0} (n+1) z^n \\
&(4) \quad\quad = \sum_{n\geq 1} T_n z^n + \sum_{n\geq 0} (n+1) z^n \\
&(5) \quad\quad = z \sum_{n\geq 1} T_n z^{n-1} + \sum_{n\geq 0} (n+1) z^n \\
&\quad\quad\quad = z \sum_{n\geq 0} T_{n+1} z^n + \sum_{n\geq 0} (n+1) z^n \\
&(6) \quad\quad = zA(z) + \frac{1}{(1-z)^2}
\end{aligned}
$$

The upshot of the work you annotated is that $A(z) = zA(z) + \frac{1}{(1-z)^2}$. Use a bit of algebra, showing your work for each step, to solve this equation for $A(z)$.

Table 2.1 : $P(n, k)$

$n\backslash k$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	2				
3	1	3	6	6			
4	1	4	12	24	24		
5	1	5	20	60	120	120	
6	1	6	30	120	360	720	720

Table 3.1 : $\binom{n}{k}$

$n\backslash k$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

Problem 6. With the usual differentiation rules of calculus applied to rational functions, find the sequences defined by the following series, where $m \geq 1$. Can you find these sequences in Table 2.1 and 3.1?

Heads up: The tables above are defined by the values of n and k, so using ℓ as your indexing variable might make it easier to see the upshot of these problems. Also, the parts of this problem are written in this order for good reason. It would be wise to consider their relationship before jumping headfirst into part (b).

(a) $\dfrac{\mathrm{d}^m}{\mathrm{d}z^m}\left[\dfrac{1}{1-z}\right]$

(b) $\dfrac{1}{(1-z)^m}$

1.3 Hey! Huh? Wait! What are these good for?

Though the above title is written to the tune of War (What Is It Good For?) by Edwin Starr, generating functions are good for a great many things, very much **not** absolutely nothing.

Definition 1.4 (Integer partition)**.** An ***integer partition*** of $n \in \mathbb{N}\,(= \{0, 1, 2, \ldots\}$ here) with k parts is. . .

- A sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{N}^k$ with $\sum \lambda_i = n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$
- A sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \in \mathbb{Z}^k_{\geq 0}$ with $\sum \lambda_i = n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell \geq 0$ with exactly k nonzero entries
  (two partitions will be considered "the same" if their nonzero entries are the same)
- An equivalence class of $k-$compositions of n under the equivalence relation setting two compositions equivalent if they are reorderings of each other.
- A multiset on $\mathbb{N}$ with cardinality k whose entires (with multiplicities) sum to n.

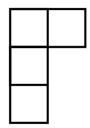
We write $\lambda \vdash n$ or $|\lambda| = n$, though the latter is harder to parse outside of context. Each λ_i is a "part" of the partition.

Example 1.5. There is a scene in the 1985 movie Clue in which two caracters argue about whether there is a bullet still in the revolver. There was one shot at Mr. Body, one (or was it two?) for the chandelier, two at the lounge door, and one for the singing telegram. What is one plus one plus two plus one? Let's see each form of this partition above.

- $(2, 1, 1, 1)$
- $(2, 1, 1, 1, 0, 0)$
- $(1, 1, 2, 1) \equiv (1, 2, 1, 1) \equiv (2, 1, 1, 1) \equiv (1, 1, 1, 2)$
- $\{1, 2, 1, 1\}$

You may come across any of these forms of partitions "in the wild" of mathematics, or you might work with any one form for some feature you want to exploit. There is a final perspective that makes partitions a classic and pretty fun set of objects to play around with.

A Young diagram for a partition λ is:
a left-justified and (under the English notation) top-justified array of boxes, with λ_i boxes in the ith row from the top. The Young diagram for example 1.5 is pictured below.



Warmup 3. Let $p(n)$ denote the number of partitions of n. In the table below, draw all partitions for n from 2 to 6, stopping after each n to record $p(n)$ make a prediction for $p(n+1)$.

n	partitions of n	$p(n)$
0	$\varnothing$	1
1	$\square$	1
2		
3		
4		
5		
6		
7		

Look at the collections of partitions for a few values of n. What do you notice about them as a whole?

Problem 7. Check your prediction for $p(7)$ ***without*** drawing all Young diagrams with 7 boxes. To do this, consider these questions:

(a) What are a few ways that we could build Young diagrams with 7 boxes from diagrams with smaller values of n "cleanly?"

(b) One way we can build larger diam is to add one or more block to the bottom of the first column or to the right of the first row of blocks. Why doesn't it matter which of these options you choose?

(c) Adding one box to the bottom a Young diagram with 6 boxes is certainly valid, and accounts for $p(6)$ many partitions of 7. What happens if we try to add two boxes to any old diagram with 5 boxes?

(d) There are *probably* two more partitions of 7 than you predicted/hope. See if you can find them all. How does the box-adding approach work to build these remaining diagrams?

At this point, you might be trying to figure out what the formula for $p(n)$ and there is some good/bad news: There is no known closed-form expression for $p(n)$. Really.

But, even though we can't write an explicit function for $p(n)$ we can still, wildly, throw generating functions at this cool family of objects and see some interesting things pop out!

1.3.1 Generating functions & partitions

Again, let $p(n)$ denote the number of partitions of n. The generating function is then

$$\sum_{n\geq 0} p(n)x^n \ .$$

Great. Hold on, this tells us nothing new. Let's try that again.

$$\sum_{n\geq 0} p(n)x^n = \prod_{i\geq 1} \frac{1}{1-x^i} \ .$$

(The pi notation above is similar to the sigma notation we have seen already, but for dealing with products instead of sums.) This is great if it is true, but it seems to come from nowhere and we don't have a reason to trust it. Yet. Let's consider the right hand side, which we can expand using our work so far in generating function world:

$$\prod_{i\geq 1} \frac{1}{1-x^i} = (1+x^1+(x^1)^2+(x^1)^3+\cdots)(1+x^2+(x^2)^2+(x^2)^3+\cdots)(1+x^3+(x^3)^2+(x^3)^3+\cdots)\cdots$$

Even though this is an infinite product of infinite-length polynomials, we can still do polynomial multiplication by FOIL-ing. Although it's more like FOIOIOIOI... Lets see just the beginning of this expansion.

$$\begin{aligned}
&(1+x^1+(x^1)^2+(x^1)^3+\cdots)(1+x^2+(x^2)^2+(x^2)^3+\cdots)(1+x^3+(x^3)^2+(x^3)^3+\cdots)\cdots \\
&= 1\cdot 1\cdot 1 + x^1\cdot 1\cdot 1 + (x^1)^2\cdot 1\cdot 1 + (x^1)^3\cdot 1\cdot 1+\cdots \\
&\quad \cdots + 1\cdot x^2\cdot 1 + x^1\cdot x^2\cdot 1 + (x^1)^2\cdot x^2\cdot 1+\cdots + 1\cdot 1\cdot x^3 + x^1\cdot 1\cdot x^3+\cdots
\end{aligned}$$

It is hard to choose a single way to begin expanding this product, but you could think about grouping like terms. In fact, we can reason that a term x^n arises in this infinite nonnegative power of x^1 times some nonnegative power times x^2 times some nonnegative power of x^3 and so on, where the powers (with multiplicities) add up to n.

Example 1.6. Here we see all the ways x^3 can arise in the infinite product.

	monomial		partition
x^3	$= (x^1)^3\cdot 1\cdot 1\cdot 1\cdots$	$\leftrightarrow$	$(1,1,1)$
	$= (x^1)^1(x^2)^1\cdot 1\cdot 1\cdots$	$\leftrightarrow$	$(2,1)$
	$= 1\cdot 1\cdot (x^3)^1\cdot 1\cdots$	$\leftrightarrow$	(3)

Warmup 4. Confirm the mechanics of the expansion above to find all the ways x^4 can arise in the infinite product.

As we can see in the example and warmup above, when x^n arises in the infinite product, the

nonnegative power of x^1	corresponds to	choosing a nonnegative number of parts 1 in $\lambda \vdash n$
nonnegative power of x^2	corresponds to	choosing a nonnegative number of parts 2 in $\lambda \vdash n$
nonnegative power of x^3	corresponds to	choosing a nonnegative number of parts 3 in $\lambda \vdash n$
$\vdots$		

Problem 8. With the understanding you've gained of the generating function for unrestricted partitions, make conjectures for generating functions for partitions. . .

(a) . . . with distinct parts

(b) . . . with odd parts (i.e., all parts are odd)

(c) . . . partitions with largest part k
Heads up: Can you really choose 1 in all terms of the product?

(d) . . . with exactly k parts

1.4 Unreasonable effectiveness of generating functions

As in many mathematical disciplines, the unreasonable effectiveness of generating functions is accessible by careful work. Here, our work begins with a friendly enough example.

Problem 9. What sequence is defined by the following generating function?

$$\frac{1}{1-5z+6z^2}$$

Hint: The sequence you find may not have an obvious, "nice" closed form. You may put the expanded form of the sequence into Online Encyclopedia of Integer Sequences (OEIS).

As in much mathematical work, our next step is to generalize.

Warmup 5. Suppose α and β are distinct nonzero real numbers. What sequences are defined by the following generating functions?

(a) $\dfrac{1}{1-\alpha z}$

(b) $\dfrac{1}{1-\beta z}$

(c) $\dfrac{1}{(1-\alpha z)(1-\beta z)}$

Sometimes in mathematics, old friends (or adversaries) make unexpected appearances. If you have not yet taken integral calculus: Know that the process in Problem 10 is called Partial Fraction Decomposition. If you have taken integral calculus: How cool is it that this has shown up? In either case, you should begin by doing one of the two always-valid algebraic manipulations: multiplying by 1 and adding 0. From there, you'll solve a system of two equations with two variables (here, A and B are the variables at hand).

Problem 10. As in Warmup 5, suppose α and β are distinct nonzero real numbers. Find constants A and B, in terms of α and β such that

$$\frac{1}{(1-\alpha z)(1-\beta z)} = \frac{A}{1-\alpha z} + \frac{B}{1-\beta z}$$

and derive another expression for the sequence you found in part (c) of Warmup 5. You may also want to reality-check your result with the example in Problem 9. (Does it matter whether α or β is 3?)

From here, the journey to seeing the unreasonable effectiveness of generating functions is made in the context of perhaps the most widely-known sequence of numbers: the Fibonacci sequence. You may not immediately see your destination, but complete each problem and trust that you'll end up at the center of the spiral.

Problem 11. Find a formula for the generating function for the Fibonacci sequence ($f_0 = 1$, $f_1 = 1$, $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$)

$$F(z) = \sum_{k \geq 0} f_k z^k = 1 + z + 2z^2 + 3z^3 + 5z^4 + 8z^5 + 13z^6 + \cdots$$

Heads up: We have already worked with a generating function involving recurrence—look back for inspiration.

Problem 12. Using the formula from Problem 11, along with the results of Problem 10, find a (non-recursive) formula for the nth Fibonacci number.

Heads up: If a quadratic doesn't factor nicely, lean on an old friend! Also, at the end, your formula probably won't be "nice", but you can check your work.

Problem 13. We can encode two-dimensional arrays of numbers with bivariate generating functions in many circumstances. Here is one example. Define the generating function $F(t, z)$ by

$$F(t, z) = \sum_{n \geq k \geq 0} a_{n,k} t^k z^n = \frac{1}{1-(1+t)z}$$

What is $a_{n,k}$?

Remark. The notation $\sum_{n \geq k \geq 0} a_{n,k} t^k z^n$ above is a way of making the double sum

$$\sum_{n \geq 0} \left(\sum_{k=0}^{n} a_{n,k} t^k \right) z^n$$

more compact. I personally prefer to write more, using the double sum, in order to see the nested relationship more clearly. You might prefer the more succinct or the more explicit notation—as long as you use it correctly, it doesn't matter which you use!

Problem 14. Use your expressions from Problems 12 and 13 to prove the following Fibonacci identity

$$f_n = \sum_{k \geq 0} \binom{n-k}{k}$$

While the Fibonacci sequence is widely known, some of the effectiveness of generating functions can be observed best via sequences even more famous to combinatorialists.

Problem 15. Let $(a_0, a_1, a_2, \ldots)$ be the sequence defined by the recurrence $a_0 = 1$ and $a_n = \sum_{i=0}^{n-1} a_i a_{n-i-1}$ for $n \geq 1$. Find an expression for the generating function of this sequence.

Heads up: This problem comes after *many* earlier problems in which we learned key tricks. You might need to use several, and if one doesn't work, try another. If you get more than one candidate, stop there (and maybe think about how you could check work using concepts from another math class...)

Problem 16. This problem explores a family of generating fucntions. Find a formula for the generating function for...

(a) ...the sequence of squares $V_2(z) = \sum_{k \geq 0} k^2 z^k = 0 + z + 4z^2 + 9z^3 + 16z^4 + 25z^5 + \cdots$

(b) ...the sequence of cubes $V_3(z) = \sum_{k \geq 0} k^3 z^k = 0 + z + 8z^2 + 27z^3 + 64z^4 + 125z^5 + \cdots$

(c) Let V_n denote the generating function for the sequence of n^{th} powers:

$$V_n(z) = \sum_{k \geq 0} k^n z^k = 0 + z + 2^n z^2 + 3^n z^3 + 4^n z^4 + 5^n z^5 + \cdots .$$

Find a formula for $V_n(z)$ in terms of $V_{n-1}(z)$.

Warmup 6. With $V_n(z)$ as in Problem 16, let $A_n(z) = (1-z)^{n+1} V_n(z)$. Do work (as seen way back in Warmup 2) to find a much nicer expression for $A_1(z)$ and $A_2(z)$, and $A_3(z)$ if you're feeling audacious. What do you notice?

Problem 17. With $A_n(z)$ as in Warmup 6, a formula for $A_n(z)$ in terms of $A_{n-1}(z)$, and use this recurrence (perhaps with the aid of a computer) to make a table of these polynomials for $n = 1, 2, \ldots, 8$. What happens when you set $z = 1$ in $A_n(z)$?

Hint: The result from part (c) of Problem 16 should definitely come into play. In light of that result, you can change your perspective and use another, much earlier result. Your formula for $A_n(z)$ will not be simple, but there is a built-in way to see whether your results make sense (watch for a pattern, young combinatorialist).