Perspectives of an Elephant

A, B, D, Easy as 1, 2, 3

Instead of starting our class with me giving you a Definition, you're going to build understanding of a "mysterious" family of objects from a bunch of different angles. Hopefully, you'll find and angle of the things we'll define eventually that you like best!

Balls & Boxes

We'll start with a collection of 3 numbered "balls" (circles) and a collection of 3 numbered "boxes" (blanks).

In the first column, place one numbered ball in each box so that each numbered ball shows up exactly once. In the second column, you'll set up a map, from blanks to balls: with two rows or columns with all 3 numbers, draw an arrow $\mapsto$ from the number of each blank to the number of the ball in that blank. (The third column is a mystery for now.)

Balls in Boxes	**As a map**	
___ ___ ___ 1 2 3		
___ ___ ___ 1 2 3		
___ ___ ___ 1 2 3		
___ ___ ___ 1 2 3		
___ ___ ___ 1 2 3		
___ ___ ___ 1 2 3		
___ ___ ___ 1 2 3		

Triangular Trials

Write 1, 2, and 3 in the corners of the equilateral triangle provided, with 1 in the top corner, 2 in the bottom-right corner, and 3 in the bottom-left corner. Flip the triangle over and write in the corners so that both sides of the paper have the same number in each corner.

We'll start by exploring the basic operations on this triangle. At first, our starting position for each operation will be the triangle that reads $1-2-3$ clockwise from the top. For each operation, your triangle should end in the same basic orientation (a clear top, bottom-right, and bottom-left).

Notation	**Operation**	**Clockwise corners**
id	Don't move the triangle.	
f_1	Hold the top corner (w/ 1 in it), and flip the triangle over.	
f_2	Hold the bottom-right corner (w/ 2 in it), and flip the triangle over.	
f_3	Hold the bottom-left corner (w/ 3 in it), and flip the triangle over.	
r_1	Rotate the triangle $2\pi/3$ radians (120°) clockwise.	
r_2	Rotate the triangle $2\pi/3$ radians (120°) counter-clockwise.	

Next, we'll start ***composing*** these operations, doing one then another then another (reading left to right), and so on, without resetting to the $1-2-3$ order between steps. As you do the flip moves, you should focus on the *location* of the corner you're holding, not what number is there.

Sequence of operations	**Clockwise corners**	
$f_3 f_3$		
$f_3 f_1$		
$f_1 f_3$		
$f_1 f_3 f_1$		
$f_3 f_1 f_3$		
$r_1 r_1$		
$r_1 r_2$		
$r_1 r_1 r_1$		
$f_1 r_1$		
$r_1 f_1$		

Matrices

We'll start with a vector $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, which we can think of as defining a point in $\mathbb{R}^3$.

Any 3×3 matrix, which has 3 rows and 3 columns, will map $\vec{x}$ to some other point in $\mathbb{R}^3$. We'll think about special kinds of matrices, which have exactly three entries 1 and all other entries 0. There should be exactly one 1 entry in each row and exactly one 1 entry in each column.

1. Write down all 3×3 matrices that fit the description above, in the first row below.

2. Go to `https://www.desmos.com/matrix` or the matrix calculator of your choice. For each of your matrices A, do the matrix multiplication $A\vec{x}$. Note: the multiplication $\vec{x}A$ won't work because $\vec{x}$ is a 3×1 matrix, having 3 rows and 1 column. In order for two matrices to multiply in the order AB, there must be the same number of *columns* in A as there are *rows* in B. Write the result of $A\vec{x}$ in the second row.

3. For each matrix A, use the matrix calculator to find A^2, A^3, and A^4. Write down the lowest power of A that gives us the ***identity matrix*** I_3, which has 1's along the main diagonal and 0's everywhere else: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Go Exploring!

For your favorite approach (or two if you're feeling really curious), explore the next level up:

- **Balls & Boxes**
  - With 3 balls & boxes:
    - Fill in the third column by writing down another map: draw an arrow $\mapsto$ from each number of ball to the number of the blank that ball is in.
    - What happens if you do the second column map and then the third column map? Third then second? You can do each of these by "stacking" the two maps on top of/right beside one another and writing a new map for the overall result.
  - With 4 balls & boxes:
    - Find all the ways to put 4 numbered balls into 4 numbered boxes. (Think about how to know when you've found them all/how you would find one you were missing.)
    - Choose a few examples of the 4-ball orderings to see whether the idea you came up about the two maps carries over to this bigger group.
    - Let s_1 be the map that sends 1 to 2, 2 to 1, and 3 and 4 to themselves. Choose at least four 4-ball orderings, and for each ordering w you choose, find the simplified map for: ws_1 (w then s_1) and s_1w (s_1 then w). What do you notice about what multiplying w on the left by s_1 or on the right by s_1 does to the resulting map?
    - Let s_2 be the map that sends 1 to itself, sends 2 to 3 and vice versa, and sends 4 to itself. Let s_3 be the map that sends 1 and 2 to themselves and swaps 3 and 4. For the four 4-ball orderings you chose, can you predict what ws_2, s_2w, ws_3, and s_3w should be? Check a few of your answers using the long-hand process you've done previously.

- **Triangular Trials**
  - With your triangle:
    * For each sequence of operations in the second table, what operation from the first table does that sequence match? Write your answers in the third column of the table.
    * Can you get every possible "Clockwise corners" order with just f_1 and f_3? What about with just r_1 and r_2? What about with just r_1 and f_1? If so, how can you get each "basic" operation from the first table? If not, what operations are you missing?
    * If we only do f_1 and f_3, what's the minimum number of moves we need to get back to the order $1-2-3$?
    * If we never do f_1 twice in a row or f_3 twice in a row, what's the maximum number of moves we can use to get back to the order $1-2-3$ for the first time?
  - With a regular tetrahedron (You'll need some imagination and maybe a pal to work with for this. Be patient.):
    * Hold the tetrahedron in a fixed position and decide on an order $1-2-3-4$ of the corners that make sense to you. Consider the reflection of the tetrahedron that swaps the vertices 1 and 2 but leaves 3 and 4 where they are (reflecting across an imaginary plane that contains the edge between 3 and 4 and cuts the edge between 1 and 2 in half). We will call that reflection s_1. There is also a reflection that swaps 2 and 3 but leaves 1 and 4 where they are (we'll call this s_2) and one that swaps 3 and 4 but leaves 1 and 2 where they are (we'll call this s_3).
    * For each sequence of two reflections, write the new ordering of corners according to the *physical locations* you decided for $1-2-3-4$.
    * Determine whether there are redundant sequences of two reflections (that give the same ordering of corners), then for each ordering of corners, choose a favorite sequence of reflections. For each unique and favorite two-reflection sequence, write the ordering of corners (again according to your physical locations of $1-2-3-4$) for three-reflection sequences beginning with that sequence. *Hint: you could work from scratch, or you could pick up where you left off.*

- **Matrices**
  - With 3×3 matrices:
    * Plot the the result of each $A\vec{x}$ in your favorite 3D visualization tool. Do these points make a 3D shape, really? What shape do they make?
    * Add the planes $x = y$, $y = z$, and $x = z$ to the visualizer. What do you notice?
    * If we delete the same entry from all of our points, we will end up with some number of points in 2D. Plot those points. What do you notice about the shape made by these new points?
    * What's the smallest collection of lines that defines the shape made by the points? (How many points should each of these lines cross through? How many points should be on either side of the line?) List and plot this smallest collection of lines.
  - With 4×4 matrices:
    * Consider the vector $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$. Write down all 4×4 matrices with exactly one 1 entry in each row and column. For each matrix A, find the lowest power of A that gives us I_4, and calculate $A\vec{x}$.
    * We can't quite plot the collection of vectors $A\vec{x}$, but using takeaways from our 3D points, consider whether there is a way to get a 3D shape from our collection of 4D points.
    * What's the smallest collection of planes that defines this shape? (You'll probably want to look at the points from various angles and then toggle the view of planes on and off so you're seeing only some of them at a time.) How many points lie on each plane? How many are on either side?